

# EXPONENTIAL GROWTH AND DECAY

Math 130 - Essentials of Calculus

30 October 2019

# EXPONENTIAL GROWTH

In the situation where a quantity changes at a constant percentage rate, we can model the situation with a *differential equation*:

$$A'(t) = k \cdot A(t)$$

which says that the rate of change in  $A$  is equal to  $k$  times  $A$ .

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which says that the rate of change in  $A$  is equal to  $k$  times  $A$ . The solution to this differential equation is given by

$$A(t) = Ce^{kt}$$

where  $C$  is the *initial value* or initial quantity.  $k$  represents the *relative growth rate*.

# EXPONENTIAL GROWTH

## EXAMPLE

*A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour, the population has increased to 420.*

- ① *Find the relative growth rate.*
- ② *Find an expression for the number of bacteria after  $t$  hours.*
- ③ *Find the number of bacteria after three hours.*
- ④ *Find the rate of growth after three hours.*
- ⑤ *When will the population reach 10,000?*

# NOW YOU TRY IT!

## EXAMPLE

*The half-life of the radioactive material cesium-137 is 30 years. Suppose we have a 100mg sample.*

- 1 *Find the relative growth rate.*
- 2 *Write a formula that gives the mass that remains after  $t$  years.*
- 3 *How much of the sample remains after 100 years?*
- 4 *After how long will only 1 mg remain?*
- 5 *At what rate is the mass decreasing after 100 years?*

# LOGISTIC GROWTH

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Sometimes a population increases exponentially at first, but then levels off when approaching its maximum population, called the *carrying capacity*, that the environmental conditions can sustain. The model for a situation like this is called a *logistic function* which has the form

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

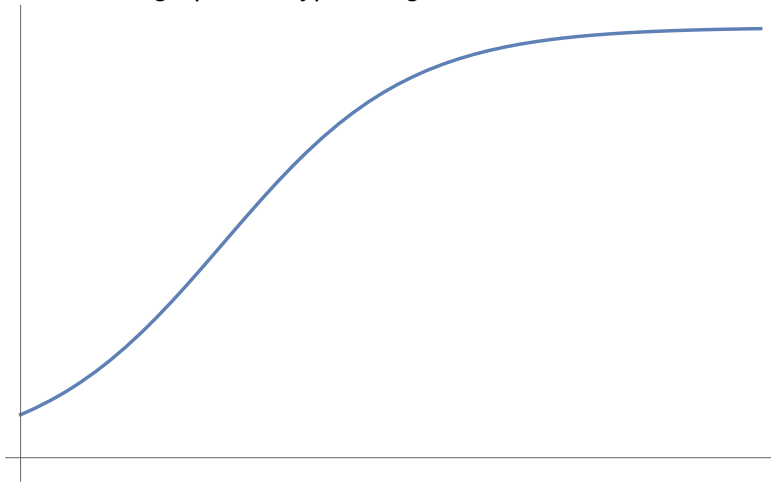
where  $M$  is the carrying capacity,  $t$  is time,  $k$  is a constant, and  $A$  is a constant given by

$$A = \frac{M - P_0}{P_0}$$

where  $P_0$  is the *initial population*.

# LOGISTIC GROWTH

Here is the graph of a typical logistic function





# LOGISTIC GROWTH

Logistic growth also satisfies a differential equation given by

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The meaning of this equation is that the rate of change of the population is proportional to the product of the population and how far the population is from the carrying capacity.

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## EXAMPLE

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*A lake is stocked with 1000 fish and the fish population is expected to follow the model*

$$P(t) = \frac{17,000}{1 + 16e^{-0.7t}}$$

*where  $t$  is the time elapsed, in years.*

- ① *What is the carrying capacity?*
- ② *What is the fish population after 2.5 years?*
- ③ *How many years are required for the fish population to reach 12,000?*
- ④ *What is the growth rate of the fish population after five years?*

## EXAMPLE

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*The number of mountain lions in a wildlife preserve is modeled by*

$$P(t) = \frac{1680}{1 + 4.2e^{-0.11t}}$$

*where  $t$  is the number of years after January 1, 2010.*

- ❶ *What is the carrying capacity? How many mountain lions are there on January 1, 2010?*
- ❷ *According to the model, what is the population after 15 years?*
- ❸ *When does the model predict that the mountain lion population will reach 1500?*
- ❹ *Compute and interpret  $P'(12)$ .*